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## Zero energy divergence of scattering cross sections in two dimensions

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**Abstract.** It has been known for a long time that any attractive two-dimensional potential with circular symmetry has a bound state. A simple generalisation of standard scattering theory to two dimensions shows that all cross sections diverge at zero energy; furthermore a  $T$  matrix theory can show that those two results are connected, that they correspond to the same kinematical singularity and that they exist for non-circular potentials.

### 1. Introduction

In all standard quantum mechanics textbooks three-dimensional theory is developed, because it is realistic; very often one-dimensional examples are treated because they are pedagogical. Two-dimensional quantum mechanics is generally ignored, and this may be the reason why some peculiarities of it seem to have been overlooked. Nevertheless it may be needed for the physics of surfaces, heterojunctions, quantum wells and even thin films.

One result has nevertheless been known for a long time: a potential well of circular symmetry always has a bound state if its space integral is negative (Landau and Lifshitz 1958); the binding energy of such a state varies as  $\exp(-|\bar{V}|^{-1})$  where  $\bar{V}$  is proportional to the space integral of the potential. Such non-analytical behaviour shows the marginality of the two-dimensional case of quantum mechanics, which has also been emphasised by the renormalisation group results on Anderson localisation (Abrahams *et al* 1979).

A formalism has been developed to describe 2D scattering events (Stern and Howard 1967, Lapidus 1982), but it does not yet seem to have been realised that the singularity on the bound state has its equivalent for scattering. The main result is the following: all cross sections diverge at zero energy, although the phase shift tangent goes normally to zero. Nevertheless, the fact that Jost functions cannot be defined in two dimensions, if it had been noticed, would have suggested this result. In fact, it will be shown in this paper that the bound state marginality and scattering divergence are closely related, and that they do not need circular symmetry.

This paper is organised as follows. In § 2, two-dimensional Green functions are computed and the 2D scattering formalism is given both for completeness and because the phase convention suggested by Green functions is different from the one used by previous authors. Then, in § 3, it is shown that, for rapidly decreasing potentials, simple equalisation of the logarithmic derivative of the radial part of the wavefunction gives the results, and in § 4, a  $T$  matrix formalism, invented by Noyes (1965), is used for demonstrating generally both the bound state and the scattering results.

## 2. Two-dimensional scattering formalism

We shall have to discuss solutions of the following Schrödinger equation:

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + V(x, y) \psi = E \psi \quad (1)$$

and to analyse them according to their symmetry. The usual spherical harmonic decomposition is replaced by a simple Fourier series and, if  $r$  and  $\theta$  are polar coordinates, we can write

$$V(r, \theta) = \sum_{-\infty}^{\infty} V_{\mu}(r) \exp(i\mu\theta) \quad \psi(r, \theta) = \sum_{-\infty}^{\infty} R_{\lambda}(r) \exp(i\lambda\theta). \quad (2a, b)$$

For a circular potential

$$V_{\mu}(r) = 0 \quad \mu \neq 0$$

the Schrödinger equation is written

$$-\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\lambda^2}{r^2} \right) R_{\lambda}(r) + V(r) R_{\lambda}(r) = E R_{\lambda}(r). \quad (3)$$

With reduced units

$$U(r) = (2m/\hbar^2) V(r) \quad (4)$$

$$k^2 = -\kappa^2 = (2m/\hbar^2) E = E' \quad (5)$$

where  $k^2$  will be used for scattering states and  $\kappa^2$  for bound states, (3) becomes

$$\frac{d^2 R_{\lambda}}{dr^2} + \frac{1}{r} \frac{dR_{\lambda}}{dr} + \left( k^2 - U(r) - \frac{\lambda^2}{r^2} \right) R_{\lambda} = 0. \quad (6)$$

If one now makes the transformation

$$R_{\lambda}(r) = \chi_{\lambda}(r) r^{-1/2}$$

the Schrödinger equation becomes

$$\frac{d^2 \chi_{\lambda}}{dr^2} + \left( k^2 - U(r) - \frac{(\lambda - \frac{1}{2})(\lambda + \frac{1}{2})}{r^2} \right) \chi_{\lambda} = 0. \quad (7)$$

Here one can see that the centrifugal term is exactly the same as in three dimensions if one makes the substitution

$$l = \lambda - \frac{1}{2}. \quad (8)$$

In the theory of Jost functions one retains the solution of (7) starting like  $r^l$  and rejects that starting like  $r^{-l-1}$ , but here, for  $\lambda = 0$  or  $l = -\frac{1}{2}$  waves, they are both the same so now we have the source of the trouble to be described for  $\sigma$  waves.

To build a scattering formalism, it is useful to start by computing the Green function of the kinetic energy operator, i.e. to solve

$$(\nabla^2 + k^2) G_k(\boldsymbol{\rho}) = \delta(\boldsymbol{\rho}) \quad (9)$$

for positive energies and

$$(\nabla^2 - \kappa^2) G_{\kappa}(\boldsymbol{\rho}) = \delta(\boldsymbol{\rho}) \quad (9')$$

for negative energies. We shall use the Hankel integrals (Watson 1922)

$$\int_0^{\infty} \frac{xJ_0(ax)}{x^2 - k^2} dx = \frac{i\pi}{2} H_0^1(ka) \quad (10)$$

and

$$\int_0^{\infty} \frac{xJ_0(ax)}{x^2 + \kappa^2} dx = K_0(\kappa a) \quad (10')$$

valid when  $a > 0$ ,  $\text{Im}(k) > 0$ ,  $\text{Re}(\kappa) > 0$ ;  $J_n$ ,  $H_n^1$  and  $K_0$  are Bessel functions, Hankel functions of the first type and Macdonald functions respectively; in fact (10) and (10') are the same formula, as can be seen from the standard relation (Nikiforov and Ouvarov 1976):

$$K_\nu(z) = \frac{1}{2}\pi \exp[i(\nu + 1)\pi/2] H_\nu^1(iz). \quad (11)$$

A Fourier transform of (9) followed by the use of (10) gives

$$G_k(\rho) = -\frac{1}{4}i H_0^1(k\rho). \quad (12)$$

The asymptotic expansion of (12) for large  $\rho$  gives

$$\begin{aligned} G_k(\rho) &\sim -\frac{i}{4} \left(\frac{2}{\pi k\rho}\right)^{1/2} \exp[i(k\rho - \pi/4)] \\ &\sim -\frac{1}{2}(2\pi k\rho)^{-1/2} \exp[i(k\rho + \pi/4)] \end{aligned} \quad (13)$$

while for small distances

$$G_k(\rho) = \frac{1}{2\pi} \ln\left(\frac{C_1 k\rho}{2}\right) - \frac{1}{4}i + \dots \quad (14)$$

where  $C_1 = e^\gamma$ ,  $\gamma$  being the Euler-Mascheroni constant. The addition formula for solutions of Bessel equations (Watson 1922) gives, if

$$\rho^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta') \quad (15)$$

$$G_k(\rho) = -\frac{1}{4}i \sum_{m=-\infty}^{\infty} H_m(kr_>) J_m(kr_<) \exp[im(\theta - \theta')] \quad (16)$$

where  $r_> = \max(r, r')$ ,  $r_< = \min(r, r')$ .

Similarly, (10') is solved by

$$G_\kappa(\rho) = -\frac{1}{2\pi} K_0(\kappa\rho). \quad (12')$$

For large  $\rho$ , it has the asymptotic expansion

$$G_\kappa(\rho) \sim -\frac{1}{2}(2\pi\kappa\rho)^{-1/2} \exp(-\kappa\rho) \quad (13')$$

and for small  $\rho$

$$G_\kappa(\rho) = \frac{1}{2\pi} \ln\left(\frac{C_1 \kappa\rho}{2}\right) + \dots \quad (14')$$

The addition formula gives here

$$G_\kappa(\rho) = -\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} K_m(\kappa r_>) I_m(\kappa r_<) \exp[im(\theta - \theta')] \quad (16')$$

where  $I_m$  is the modified Bessel function

$$I_\nu(r) = i^{-\nu} J_\nu(iz).$$

The scattering of a two-dimensional particle by a circular symmetry potential can be described by a phase-shift formalism, which will be described both for completeness and because the use of the Green function (12) changes the phase convention form what has been previously used and makes the formula more similar to those used for 3D problems. Writing the Lippmann-Schwinger equation for retarded waves, the incoming particle being directed along the  $x$  axis

$$\psi(\mathbf{r}) = \exp(ikx) - \frac{1}{4}i \int d^2\mathbf{r}' H_0^1(k|\mathbf{r} - \mathbf{r}'|) U(\mathbf{r}')\psi(\mathbf{r}') \tag{17}$$

and using the asymptotic expansion (13), one has for large  $r$

$$\psi(\mathbf{r}) \sim \exp(ikx) - \frac{i}{4} \int d^2\mathbf{r}' \left( \frac{2}{\pi k|\mathbf{r} - \mathbf{r}'|} \right)^{1/2} \exp[i(k|\mathbf{r} - \mathbf{r}'| - \pi/4)] U(\mathbf{r}')\psi(\mathbf{r}')$$

one defines as usual  $\mathbf{k}'$  as the vector of modulus  $k$  parallel to  $\mathbf{r}$ , and one obtains

$$\psi(\mathbf{r}) \sim \exp(ikx) - \frac{i}{\sqrt{2}} \exp(-i\pi/4) \exp(ikr) \frac{1}{2(2\pi k)^{1/2}} \int d^2\mathbf{r}' \exp(-i\mathbf{k}' \cdot \mathbf{r}') U(\mathbf{r}')\psi(\mathbf{r}'). \tag{18}$$

If one defines the scattering amplitude  $f(\theta)$  by the boundary condition

$$\psi(r, \theta) \sim \exp(ikr \cos \theta) + f(\theta)r^{-1/2} \exp[i(kr + \pi/4)] \tag{19}$$

one obtains the formula

$$f(\theta) = -\frac{1}{2(2\pi k)^{1/2}} \int d^2\mathbf{r}' \exp(-i\mathbf{k}' \cdot \mathbf{r}') U(\mathbf{r}')\psi(\mathbf{r}'). \tag{20}$$

If the potential is of circular symmetry,  $\psi(\mathbf{r})$  can be taken as the sum of solutions for different angular momentum  $\lambda$ :

$$\psi(r, \theta) = \sum_{\lambda=-\infty}^{\infty} i^\lambda R_\lambda(\theta) \exp(i\lambda\theta) \tag{21}$$

where each  $R_\lambda$  verifies the Schrödinger equation (6). When  $r$  is big enough for  $U(r)$  to be negligible, (6) becomes the standard Bessel equations whose solutions are the Bessel and Neumann functions of asymptotic expansion

$$J_\lambda(kr) \sim (2/\pi kr)^{1/2} \cos(kr - \frac{1}{2}\pi\lambda - \frac{1}{4}\pi)$$

$$N_\lambda(kr) \sim (2/\pi kr)^{1/2} \sin(kr - \frac{1}{2}\pi\lambda - \frac{1}{4}\pi).$$

One writes for  $R_\lambda(r)$  the following expansion:

$$\begin{aligned} R_\lambda(r) &= A_\lambda (\cos \delta_\lambda J_\lambda(kr) - \sin \delta_\lambda N_\lambda(kr)) \\ &\sim A_\lambda (2/\pi kr)^{1/2} \cos(kr - \frac{1}{2}\pi\lambda - \frac{1}{4}\pi + \delta_\lambda) \end{aligned} \tag{22}$$

and for the incident wave the expansion

$$\begin{aligned} \exp(ikr \cos \theta) &= \sum_{\lambda} i^\lambda J_\lambda(kr) \exp(i\lambda\theta) \\ &\sim (2/\pi kr)^{1/2} \sum_{\lambda} i^\lambda \cos(kr - \frac{1}{2}\pi\lambda - \frac{1}{4}\pi) \exp(i\lambda\theta). \end{aligned} \tag{23}$$

It remains to introduce (22) and (23) in (19), to use the  $\exp(-ikr)$  term to obtain

$$A_\lambda = \exp(i\delta_\lambda) \quad (24)$$

and the  $\exp(ikr)$  term to obtain

$$f(\theta) = (2/\pi k)^{1/2} \sum_\lambda \exp(i\delta_\lambda) \sin \delta_\lambda \exp(i\lambda\theta). \quad (25)$$

Now we have the differential cross section

$$\sigma(\theta) = |f(\theta)|^2 = \frac{2}{\pi k} \left| \sum_\lambda \exp(i\delta_\lambda) \sin \delta_\lambda \exp(i\lambda\theta) \right|^2 \quad (26)$$

and by integration over  $\theta$ , the total cross section

$$\sigma = \frac{4}{k} \sum_\lambda \sin^2 \delta_\lambda. \quad (27)$$

As is natural in a 2D system, the cross section is a length. From (25) and (27), one obtains the 2D form of the optical theorem

$$\sigma = (8\pi/k)^{1/2} \text{Im} f(0)$$

in a form slightly different from that given by Stern and Howard (1967).

### 3. Calculation of phase shifts

To compute phase shifts and to study their variation with energy at low  $k$ , it is a simple matter now to proceed as in three-dimensional scattering (Schiff 1968). One assumes that for  $r > a$ , where  $a$  is some distance,  $U(r)$  is negligible, so the solution of (6) is (22). On the other hand, for  $r < a$  there is one and only one solution of (6) which behaves regularly at  $r = 0$ , as long  $U(r)$  does not have too singular a behaviour (Simon 1976). Let us denote by  $R_{i\lambda}(r)$  that regular solution. The phase shift  $\delta_\lambda$  is obtained by equating the logarithmic derivatives of the internal and external solutions at  $r = a$ . So one obtains

$$\frac{R'_{i\lambda}(a)}{R_{i\lambda}(a)} = k \frac{\cos \delta_\lambda J'_\lambda(ka) - \sin \delta_\lambda N'_\lambda(ka)}{\cos \delta_\lambda J_\lambda(ka) - \sin \delta_\lambda N_\lambda(ka)}$$

i.e.

$$\tan \delta_\lambda = \frac{kR_{i\lambda}(a)J'_\lambda(ka) - R'_{i\lambda}(a)J_\lambda(ka)}{kR_{i\lambda}(a)N'_\lambda(ka) - R'_{i\lambda}(a)N_\lambda(ka)}. \quad (28)$$

For small energies,  $ka \ll 1$ , one may replace the Bessel and Neumann functions of (28), as well as their derivatives, by the first term of their series expansion

$$J_\lambda(x) = \frac{1}{\lambda!} \left(\frac{x}{2}\right)^\lambda + \dots$$

$$N_\lambda(x) = -\frac{(\lambda-1)!}{\pi} \left(\frac{2}{x}\right)^\lambda + \dots$$

for  $\lambda \neq 0$ , and

$$J_0(x) = 1 \quad J'_0(x) = -\frac{1}{2}x$$

$$N_0(x) = \frac{2}{\pi} \ln \left( \frac{C_1 x}{2} \right).$$

One obtains immediately for  $\lambda \neq 0$

$$\tan \delta_\lambda = -\frac{\pi}{\lambda!(\lambda-1)!} \frac{\lambda R_{i\lambda}(a) - a R'_{i\lambda}(a)}{\lambda R_{i\lambda}(a) + a R'_{i\lambda}(a)} \left( \frac{ka}{2} \right)^{2\lambda}. \quad (29)$$

One sees in (29) that for small  $k$ ,  $\delta_\lambda(k)$  behaves like  $k^{2\lambda}$ , and this is nothing less than the standard result for 3D where  $\delta_l \propto k^{2l+1}$ , if one remembers the correspondence law (8). However, for  $\lambda = 0$ , one gets

$$\tan \delta_0 = \frac{\pi}{2} \frac{R'_{i0}(a)}{R'_{i0}(a) \ln(C_1 ka/2) - R_{i0}(a)/a}. \quad (30)$$

In this formula it is clear that, when  $k \rightarrow 0$

$$\lim_{k \rightarrow 0} \tan \delta_0 = 0$$

but this is not enough to make the cross section finite. If one writes, from (30),

$$\delta_0 \sim \frac{\pi}{2} \left( \ln \frac{C_1 ka}{2} \right)^{-1}$$

by insertion into (27), one obtains

$$\sigma = \frac{\pi^2}{k[\ln(C_1 ka/2)]^2} \rightarrow \infty \quad (31)$$

which means that the scattering cross section for a potential of finite range diverges when the energy of the incident particle goes to zero. Of course, the above calculation is not a true demonstration valid for any potential, but one can hardly imagine a mechanism where the extension of the range of the potential would decrease the cross section.

The method used to find (31) is nothing less than the one used by Landau and Lifshitz to obtain the result quoted in the introduction. They fit the logarithmic derivative of the inner and outer solution for the ( $\lambda = 0$ ) Schrödinger equation at a point  $a$  such that  $U(r) = 0$  for  $r > a$ . Using the asymptotic expression of  $K_0(\kappa r)$ , they obtain

$$A = \ln(C_1 \kappa a/2) S(a) \quad (32)$$

where the notation

$$S(x) = \int_0^x U(r) r dr \quad (33)$$

has been introduced.

Landau and Lifshitz now obtain the quoted result by replacing  $S(a)$  by  $S(\infty)$  in (32), which is consistent with the assumption that  $U(r)$  is zero for  $r > a$ .

#### 4. The $T$ matrix formalism

Both results (31) and (33), about scattering and bound states, should be related: by a Breit-Wigner mechanism, the bound state at roughly zero negative energy is responsible for the diverging cross section at zero positive energy. That will be shown by using the Noyes variant of the  $T$  matrix expansion in powers of the strength of the potential (Noyes 1965). It has already been used by Patil (1980) to demonstrate the bound-state result for circular potential.

We shall use, for the Green operator computed above in the  $\mathbf{r}$  representation, the notation

$$G_E(\mathbf{k}_1) = \langle \mathbf{k}_1 | G_0(E' + i\varepsilon) | \mathbf{k}_1 \rangle = \frac{1}{E' + i\varepsilon - k_1^2}$$

and start from the Lippmann-Schwinger equation for the  $T$  matrix:

$$T(E') = gU + gUG_E(E')T(E') \quad (34)$$

when the potential energy operator has been multiplied by  $g$ , a coupling factor put equal to 1 at the end of the calculations.

Now the Noyes method consists in writing the matrix elements of  $T$  as a diagonal element multiplied by an off-diagonal coefficient, i.e.

$$\langle \mathbf{k} | T(E') | \mathbf{q} \rangle = h(E'; \mathbf{k}, \mathbf{q}) \langle \mathbf{q} | T(E') | \mathbf{q} \rangle. \quad (35)$$

Introducing (35) into (34) one obtains from the diagonal part of (34)

$$\langle \mathbf{q} | T(E') | \mathbf{q} \rangle \left( 1 - g \sum_{\mathbf{k}} \frac{\langle \mathbf{q} | U | \mathbf{k} \rangle h(E'; \mathbf{k}, \mathbf{q})}{E' + i\varepsilon - k^2} \right) = g \langle \mathbf{q} | U | \mathbf{q} \rangle. \quad (36)$$

This equation introduced into the off-diagonal part of (34) gives the Fredholm equation for  $h$ , i.e.

$$h(E'; \mathbf{k}, \mathbf{q}) = \frac{\langle \mathbf{k} | U | \mathbf{q} \rangle}{\langle \mathbf{q} | U | \mathbf{q} \rangle} + g \sum_{\mathbf{k}_1} \left( \langle \mathbf{k} | U | \mathbf{k}_1 \rangle - \frac{\langle \mathbf{k} | U | \mathbf{q} \rangle \langle \mathbf{q} | U | \mathbf{k}_1 \rangle}{\langle \mathbf{q} | U | \mathbf{q} \rangle} \right) \frac{1}{E' + i\varepsilon - k_1^2} h(E'; \mathbf{k}_1, \mathbf{q}). \quad (37)$$

This equation gives a natural expansion of  $h$  in powers of  $g$  and thus allows a calculation of  $T$  as a function of  $g$ , which is not the Born expansion, so it is able to be used near bound states. Here we shall only need to write (36) in the form

$$\langle \mathbf{q} | T(E') | \mathbf{q} \rangle = g \langle \mathbf{q} | U | \mathbf{q} \rangle / D_g(E') \quad (38)$$

and to keep the first order in  $g$  for  $h$ , i.e. to keep only the first two terms in

$$D_g(E') = 1 - \frac{g}{\langle \mathbf{q} | U | \mathbf{q} \rangle} \sum_{\mathbf{k}} \frac{\langle \mathbf{q} | U | \mathbf{k} \rangle \langle \mathbf{k} | U | \mathbf{q} \rangle}{E' + i\varepsilon - k^2} + \dots \quad (39)$$

To find a bound state, one must look for a pole of the  $S$  or of the  $T$  matrix, i.e. a zero of  $D_g(E)$ . To find it, one must compute the integral

$$\begin{aligned} I(E', \mathbf{q}) &= \sum_{\mathbf{k}} \langle \mathbf{q} | U | \mathbf{k} \rangle \langle \mathbf{k} | G_0(E' + i\varepsilon) | \mathbf{k} \rangle \langle \mathbf{k} | U | \mathbf{q} \rangle \\ &= \langle \mathbf{q} | U G_0(E' + i\varepsilon) U | \mathbf{q} \rangle \\ &= \int d^2\mathbf{r}_1 d^2\mathbf{r}_2 \exp(-i\mathbf{q} \cdot \mathbf{r}_1) U(\mathbf{r}_1) G_0(E'; |\mathbf{r}_1 - \mathbf{r}_2|) U(\mathbf{r}_2) \exp(i\mathbf{q} \cdot \mathbf{r}_2). \end{aligned} \quad (40)$$



With the use of the addition formula (16'), and of

$$\begin{aligned} \exp[i\mathbf{q} \cdot (\mathbf{r}_1 - \mathbf{r}_2)] &= \sum_{n=-\infty}^{\infty} i^n J_n(q|\mathbf{r}_1 - \mathbf{r}_2|) \exp[in(\theta_1 - \theta_2)] \\ &= \sum_{n=-\infty}^{\infty} i^n \sum_{n'=-\infty}^{\infty} J_{n+n'}(qr_>) J_{n'}(qr_<) \exp[in'(\theta_1 - \theta_2)] \end{aligned}$$

one finds, if the potential has circular symmetry

$$\begin{aligned} I(\mathbf{q}) &= -4\pi \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} i^n \int_0^{\infty} r_1 dr_1 J_{n+m}(qr_1) U(r_1) K_m(\kappa r_1) \\ &\quad \times \int_0^{r_1} r_2 dr_2 J_m(qr_2) U(r_2) I_m(\kappa r_2) \end{aligned} \tag{41}$$

and we can, as we are interested only in small potentials and small negative energies, replace the modified Bessel functions and the Macdonald functions by the first terms of their expansions. For  $m \neq 0$ , the  $\kappa^m$  term of the  $r_2$  integral coming from  $I_m$  will be compensated by a  $\kappa^{-m}$  term coming from  $K_m$  in the  $r_1$  integral. The only term with a  $\kappa$  dependence comes from  $m = 0$ , which has a logarithmic dependence as usual. Furthermore for  $m \neq 0$ , the  $r_2^m$  term coming from  $I_m$  lowers the  $r_2$  integral for small  $r_2$  and the  $r_1^{-m}$  term coming from  $K_m$  lowers the  $r_1$  integral for large  $r_1$ . So one is justified in keeping the only diverging term, the  $m = 0$  one, and, with the notation (33), one can write

$$I(\mathbf{q}) = 4 \int_0^{\infty} r dr \exp(iqr) U(r) S(r) \ln(C_1 \kappa r/2).$$

One can set  $q = 0$ , as the bound-state condition does not need to be  $q$  dependent, and compute  $I(0)$ . Writing  $rU(r) = S'(r)$ , integrating by parts and neglecting the term without  $\ln \kappa a$ , one writes

$$I(0) = 2\pi S(a)^2 \ln(C_1 \kappa a/2) \tag{42}$$

which is to be introduced in the expression (39) of  $D_\lambda$ . As

$$\langle \mathbf{q} | U | \mathbf{q} \rangle = 2\pi \int_0^{\infty} U(r) r dr = 2\pi S(\infty) = 2\pi S(a) \tag{43}$$

the bound-state condition becomes

$$D(E) = 1 - S(a) \ln(C_1 \kappa a/2) = 0 \tag{44}$$

i.e. the Landau and Lifshitz result (32).

The same calculation can be done for positive energies, using the Green function (12) for positive energies. The algebra is the same,  $K_m$  being replaced by  $\frac{1}{2}i\pi H_m^1$ ,  $I_m$  by  $J_m$ . One gets at the end

$$I(0) = 2\pi S(a)^2 \ln(C_1 \kappa a/2) \tag{45}$$

the analytic continuation of (32). For low energy, small  $k$ , this term is the only one to be kept in  $D_g(E)$  and taking (43) into account, (38) becomes

$$\langle \mathbf{q} | T(E) | \mathbf{q} \rangle = -\frac{2\pi}{\ln(C_1 \kappa a/2)}. \tag{46}$$

Now the ratio of off-diagonal to diagonal terms of the  $T$  matrix is, when keeping only the first term in (37), proportional to

$$\int_0^\infty d^2r \exp[i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}] U(\mathbf{r})$$

where both phase factors are equal to unity in the area where  $U$  is not negligible, because we are interested only in low enough energies. Of course, this means that at low energy only  $\sigma$  scattering plays a role.

As (20) can be written

$$f(\theta) = -\frac{1}{2(2\pi k)^{1/2}} \langle \mathbf{k}' | T(E) | \mathbf{k} \rangle$$

it is now straightforward by using (45) to compute

$$\sigma = \int_0^{2\pi} |f(\theta)|^2 d\theta = \frac{\pi^2}{k[\ln(C_1 ka/2)]^2}$$

which is the result (31), already obtained more simply.

This  $T$  matrix method can be used to generalise those results to non-circular potentials. Starting from (40), one writes

$$U(\mathbf{r}) = U(r, \theta) = \sum_{\mu=-\infty}^{\infty} U_\mu(r) \exp(i\mu\theta) \quad (47)$$

which is nothing else than (2a) in reduced units. For bound states,  $E < 0$ , one gets, instead of (41), the expressions

$$\begin{aligned} I(\mathbf{q}) &= -\frac{1}{\pi} \sum_{\substack{m, n, n' \\ \mu, \mu'}} \int_0^\infty r_1 dr_1 J_{n+n'}(qr_1) U_\mu(r_1) K_m(\kappa r_1) \\ &\quad \times \int_0^{r_1} r_2 dr_2 J_n(qr_2) U_{\mu'}(r_2) I_m(\kappa r_2) \\ &\quad \times \int_0^{2\pi} \exp[i(m + \mu - n')\theta_1] d\theta_1 \int_0^{2\pi} \exp[i(n' - \mu' - m)\theta_2] d\theta_2 \\ &= -4\pi \sum_{m, n, \mu} \int_0^\infty r_1 dr_1 J_{(n+m+\mu)}(qr_1) U_\mu(r_1) K_m(\kappa r_1) \\ &\quad \times \int_0^{r_1} r_2 dr_2 J_{m+\mu}(qr_2) U_\mu^*(r_2) I_m(\kappa r_2) \end{aligned}$$

and in this expression one can see that for small  $\kappa$  the  $\kappa^{-m}$  power from  $K_m$  and the  $\kappa^m$  power from  $I_m$  compensate. The only  $\kappa$  dependence will come from the  $K_0$  term and will be logarithmic. As for the circularly symmetric case, replacement of  $\kappa$  by  $k$ , of  $K_m$  by  $\frac{1}{2}i\pi H_m^1$  and of  $I_m$  by  $J_m$  conserves the logarithmic singularity. So both the bound-state result and the scattering singularity do not need the circular symmetry to be observed.

## 5. Conclusion

The main applications of this result are in situations where the energy is very small, i.e. to low-temperature systems. One is the behaviour of two hydrogen atoms trapped on a He film (Bashkin 1980, Edwards 1982, Papoular 1983). The existence of a bound state for any small attractive potential is assumed to enhance the recombination of atomic hydrogen as observed by Silvera and Walraven (1981). A generalisation of the  $T$  matrix calculation to inelastic processes could explain the enhanced recombination; the divergent cross section should have an enhancing effect, even if, as has been suggested by Edwards, the interaction potentials were not attractive.

A second application is rather formal, being the singularity of the conductivity behaviour in disordered two-dimensional systems. It is well known (Abrahams *et al* 1979) that  $d = 2$  is the critical dimension for localisation and this is a result of a simple renormalisation group argument. It has been shown that this problem of localisation is equivalent to solving a Bethe-Salpeter equation in the electron-electron channel around  $k + k' = 0$  (Altshuler *et al* 1982, Economou and Soukoulis 1983) and this is equivalent to solving a one-particle problem; the fact that the potential is to be determined self-consistently from its solution (Vollhardt and Wölfle 1980) does not change the formal results derived above.

One extension of the  $T$  matrix calculation is to be done in further work. It is to introduce internal degrees of freedom of the incident particle, either to take into account spin effects and spin-orbit coupling which is known to have an important effect in localisation problems (Hikami *et al* 1980), or to describe the fact that there may be some different levels due to the potential perpendicular to the 2D systems described above (Ando *et al* 1982). In particular, one should probe the Bashkin criterion for the application of 2D bound-state theory (Bashkin 1980) that the radius of the bound state should be bigger than the physical width of the slab and find the analogous criterion for scattering.

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## References

- Abrahams E, Anderson P W, Licciardello D C and Ramakrishnan T V 1979 *Phys. Rev. Lett.* **42** 673
- Altshuler B L, Aronov A G, Khmel'nitskii D E and Larkin A I 1982 *Quantum Theory of Solids* ed I M Lifshits (Moscow: Mir) p 130
- Ando T, Fowler A B and Stern F 1982 *Rev. Mod. Phys.* **54** 437
- Bashkin E P 1980 *Sov. Phys.-JETP* **51** 181
- Economou E N and Soukoulis C M 1983 *Phys. Rev. B* **28** 1093
- Edwards D O 1982 *Physica B+C* 1531
- Hikami S, Larkin A I and Nagaoka Y 1980 *Prog. Theor. Phys.* **63** 707
- Landau L D and Lifshitz E M 1958 *Quantum Mechanics* (Oxford: Pergamon) p 156
- Lapidus I R 1982 *Am. J. Phys.* **50** 45
- Nikiforov A and Ouarov V 1976 *Eléments de la théorie des fonctions spéciales* (Moscow: Mir)

- Noyes H P 1965 *Phys. Rev. Lett.* **15** 538  
Papoular M 1983 *J. Low Temp. Phys.* **50** 253  
Patil S H 1980 *Phys. Rev. A* **22** 2400  
Schiff L I 1968 *Quantum Mechanics* 3rd edn (New York: McGraw-Hill) p 121  
Silvera I F and Walraven J T M 1980 *Phys. Rev. Lett.* **45** 1268  
Simon B 1976 *Ann. Phys., NY* **97** 279  
Stern F and Howard W E 1967 *Phys. Rev.* **163** 816  
Vollhardt D and Wölfle P 1980 *Phys. Rev. B* **22** 4666  
Watson G N 1922 *Theory of Bessel Functions* (Cambridge: Cambridge University Press)